High Order Fourier-Spectral Solutions to Self Adjoint Elliptic Equations

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Abstract. We develop a High Order Fourier solver for nonseparable, selfadjoint elliptic equations with variable (diffusion) coefficients. The solution of an auxiliary constant coefficient equation, serves in a transformation of the dependent variable. There results a "modified Helmholtz" elliptic equation with almost constant coefficients. The small deviations from constancy are treated as correction terms. We developed a highly accurate, fast, Fourier-spectral algorithm to solve such constant coefficient equations. A small number of correction steps is required in order to achieve very high accuracy. This is achieved by optimization of the coefficients in the auxiliary equation. For given coefficients the approximation error becomes smaller as the domain decreases. A highly parallelizable hierarchical procedure allows a decomposition into smaller subdomains where the solution is efficiently computed. This step is followed by hierarchical matching to reconstruct the global solution. Numerical experiments illustrate the high accuracy of the approach even at coarse resolutions.

1 Introduction

Variable coefficient elliptic equations are ubiquitous in many scientific and engineering applications the most important case being that of the self-adjoint operator appearing for example in diffusion processes in non uniform media. Many repeated solutions of such problems are required when solving variable coefficient or non linear time dependent problems by implicit marching methods.

Application of high-order (pseudo) spectral methods, which are based on global expansions into orthogonal polynomials (Chebyshev or Legendre polynomials), to the solution of elliptic equations, results in full (dense) matrix problems. The spectral element method allows for some sparsity. On the other hand the Fourier spectral method for the solution of the Poisson equation gives rise to diagonal matrices and has an exponential rate of convergence but looses accuracy for non-periodic boundary conditions due to the Gibbs phenomenon.

Our method to resolve the Gibbs phenomenon represents the RHS as a sum of a smooth periodic function and another function which can be integrated analytically. This approach is sometimes called "subtraction".

The subtraction technique for the reduction of the Gibbs phenomenon in the Fourier series solution of the Poisson equation goes back to Sköllermo [2] who considered,

$$\Delta u = f \tag{1}$$

in the rectangle $[0,1] \times [0,1]$ with non periodic boundary conditions. We note that the subtraction algorithm in [2] was of limited applicability. We develop in section 4 a high order generalization for the case of the modified Helmholtz equation. The Poisson equation case is just a particular case.

The subtraction technique (in the physical space) has the following advantages:

a) After subtraction, the Fast Fourier Transform can be applied to the remaining part of RHS with a high convergence rate.
b) The algorithm preserves the diagonal representation of the Laplace operator.
c) . The computation of the subtraction functions inexpensive.

In the framework of the present paper we solve the elliptic equation:

$$\nabla \cdot (a(x,y)\ \nabla u(x,y)) - c(x,y)u(x,y) = f(x,y), \quad (x,y) \in \mathbf{D}, \tag{2}$$

where $\mathbf{D}$ is a rectangular domain, with the Dirichlet boundary conditions

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\mathbf{D}. \tag{3}$$

We assume $a(x,y) > 0$ for any $(x,y) \in \mathbf{D}$.

1. We develop first a fast direct algorithm for the solution of Eq. (2) for any function $a(x,y)$, such that $a(x,y)^{1/2}$ is equal to the solution $w(x,y)$ of a certain, appropriately chosen, constant coefficient equation (see below). The algorithm is based on our improvement of the fast direct solver of [**?**] and a transformation described in [3].
2. If $a(x,y)^{1/2}$ is not equal to $w(x,y)$, we substitute $w(x,y)^2$ for $a(x,y)$ and transfer the difference to the right hand side. The solution is found in a short sequence of correction steps.
3. An adaptive hierarchical domain decomposition approach allows improved approximation for any function $a(x,y)$.

2 Outline of the Algorithm

Following [3] we make the following change of variable in Eq. (2):

$$w(x,y) = a(x,y)^{1/2}u(x,y), \tag{4}$$

then Eq. (2) takes the form

$$\Delta w(x,y) - p(x,y)w(x,y) = q(x,y), \tag{5}$$

where

$$p(x,y) = \Delta(a(x,y)^{1/2}) \cdot a(x,y)^{-1/2} + c(x,y) \cdot a(x,y)^{-1},$$
$$q(x,y) = f(x,y) \cdot a(x,y)^{-1/2}. \tag{6}$$

If $p(x,y)$ happens to be a constant we have achieved a reduction to a constant coefficient case. As $a(x,y)$ and $c(x,y)$ are prescribed in the formulation of the problem we have no control over $p(x,y)$, nevertheless we will show that a constant approximation to $p(x,y)$ is achievable. We note that in the particular case where $a(x,y)^{1/2}$ is a harmonic function, Eq. (12) becomes a Poisson equation for w:

$$\Delta w(x,y) = q(x,y) \tag{7}$$

This leads to a fast direct algorithm for the numerical solution of Eq. (2), where $a(x,y)^{1/2}$ is a harmonic function.

Algorithm A

1. Using the modified spectral subtractional algorithm which was described in the introduction, we solve Eq. (7) with the boundary conditions $\tilde{g}(x,y) = a(x,y)^{1/2} \cdot g(x,y)$.
2. The solution of Eq. (2) is $u(x,y) = w(x,y) \cdot a(x,y)^{-1/2}$.

Let us now consider the case where $a(x,y)^{1/2}$ is not exactly harmonic but can be well approximated by a harmonic function $\widetilde{a}(x,y)^{1/2}$. This means that the difference

$$\varepsilon(x,y) = a(x,y) - \widetilde{a}(x,y) \tag{8}$$

is small. Denote by u_0 the solution of the equation where $a(x,y)$ is replaced by $\widetilde{a}(x,y)$. Then the following correction procedure can be used:

$$\nabla \cdot \big(\widetilde{a}(x,y)\ \nabla(u^1 - u_0)\big) = -\nabla \cdot (\varepsilon(x,y)\ \nabla u_0) \tag{9}$$

$$\nabla \cdot \big(\widetilde{a}(x,y)\ \nabla(u^{n+1} - u_0)\big) = -\nabla \cdot (\varepsilon(x,y)\ \nabla u^n)\,, \quad n \geq 1. \tag{10}$$

Here, u^n is the corrected solution after n correction steps. Suppose $\|\varepsilon\| \leq s\|a\|$ in a certain Sobolev semi-norm, where s is small It follows that the error decreases according to:

$$\|u^{n+1} - u\| \leq s\|u^n - u\| \tag{11}$$

3 The auxiliary equation

If p in Eq. (6) is not zero but a constant (larger then the first eigenvalue of the Laplacian) we have an elliptic constant coefficient partial differential equation of Helmholtz or modified Helmholtz type. Such equations can be easily solved by the subtraction technique as illustrated in section 4. By assumption, $a(x,y)^{1/2}$ is positive and does not vanish. Consider for example a region R, the values of

$a(x,y)^{1/2}$ on its boundary are positive which is tantamount to positive Dirichlet boundary conditions for our approximation which should satisfy also the equation:

$$\Delta w(x,y) - Pw(x,y) = c(x,y)a(x,y)^{-1/2}, \tag{12}$$

where P is a constant to be chosen so that $w(x,y)$ gives the best approximation to $a(x,y)^{1/2}$. If $a(x,y)^{1/2}$ is constant on the boundaries and dome shaped, and c(x,y) vanishes, the harmonic approximation will be a horizontal plane. On the other hand a negative P will give rise to a dome shaped approximation, and P can be chosen so that the function $w(x,y)$ will match the height of the dome. As we take more negative P (but larger then the lowest eigenvalue of the Laplacian) we get higher and higher domes. Conversely, if $a(x,y)^{1/2}$ is bowl shaped, a positive P will give rise to deeper and deeper bowls. For large P we will get values close to zero in most of the interior of R.

4 Solution of Modified Helmholtz Equation in a box

In this section, we will describe a method for the solution of Modified Helmholtz equation with arbitrary order accuracy. We will start with an algorithm of $O(N^{-4})$ order of accuracy, then we construct the algorithm for $O(N^{-6})$ and generalize it to the arbitrary order of accuracy.

4.1 Problem Formulation

We are interested in the solution of the two-dimensional Modified Helmholtz (MH) equation in the rectangular region $\Omega = [0\ 1] \times [0\ 1]$ with Dirichlet boundary conditions.

$$\begin{cases} \triangle u(x,y) - k^2 u(x,y) = f(x,y) & \text{in } \Omega \\ u(x,y) = \Phi(x,y) & \text{on } \partial\Omega \end{cases} \tag{13}$$

The boundary functions

$$\phi_1(x) \triangleq \Phi(x,0), \quad \phi_3(x) \triangleq \Phi(x,1)$$
$$\phi_2(y) \triangleq \Phi(0,y), \quad \phi_4(y) \triangleq \Phi(1,y)$$

are assumed to be smooth and continuous at the corners . In addition, $f(x, y)$ is supposed to be known on $\partial\Omega$. We introduce the following notations:

$$f^{(p)}(x) \triangleq \frac{\partial^p f(x)}{\partial x^p}, \quad f^{(p,q)}(x, y) \triangleq \frac{\partial^{p+q} f(x, y)}{\partial x^p \partial y^q}$$

$$\text{Vandermonde}(\lambda_1, \lambda_2, \ldots, \lambda_n) \triangleq \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix}$$

$$\mathcal{H}_M \triangleq \triangle - k^2$$

Let $I = \{1, 2, 3, 4\}$ be an index set of corner points or edges. Denote by $p_j,\ j \in I$ the four corner points of $\partial\Omega$:

$$p_1 = (0, 0),\ p_2 = (0, 1),\ p_3 = (1, 1),\ p_4 = (1, 0)$$

and by $E_j,\ j \in I$ the four edges of $\partial\Omega$:

$$\begin{aligned} E_1 &= \{(x, y)|y = 0\}, \quad E_2 = \{(x, y)|x = 0\}, \\ E_3 &= \{(x, y)|y = 1\}, \quad E_4 = \{(x, y)|x = 1\}, \end{aligned}$$

and define

$$\partial\Omega_C \triangleq \{p_j \,|\, j \in I\}, \quad \partial\Omega_E \triangleq \partial\Omega \setminus \partial\Omega_C$$

4.2 Constructions of Auxiliary Function

In order to apply the subtraction technique, we construct a family of functions $q_{2r}(x), r \geqslant 0$ with the following property:

$$\begin{aligned} & q_{2r}(1) = 1, \quad q_{2r}(0) = 0 && \text{if} \quad r = 0 \\ & \begin{cases} q_{2r}^{(2s)}(0) = 0, \quad q_{2r}^{(2s)}(1) = 0, \quad 0 \leqslant s \leqslant r - 1 \\ q_{2r}^{(2r)}(0) = 1, \quad q_{2r}^{(2r)}(1) = 0 \end{cases} && \text{if} \quad r \geqslant 1 \end{aligned} \tag{14}$$

We look for a function $q_{2r}(x)$ as the linear combination

$$q_{2r}(x) = \sum_{i=1}^{r+1} \alpha_{2r,i} \frac{\sinh(\lambda_{2r,i}\,(1 - x))}{\sinh(\lambda_{2r,i})}, \qquad \text{where} \quad \forall i:\ \lambda_{2r,i} > 0 \tag{15}$$

Lemma 1. *For any $r \geqslant 0$ we can find constants $\alpha_i \in \mathbb{R}$ and $0 < \lambda_i \in \mathbb{R}$ such that the function $q_{2r}(x)$ takes form of (15)*

Lemma 2. *Let $\lambda, \mu > 0$ and define $f(x,y)$ as follows*

$$f(x,y) \triangleq \frac{\sinh(\lambda\,(1-x))}{\sinh(\lambda)}\,\frac{\sinh(\mu\,(1-y))}{\sinh(\mu)} \tag{16}$$

If in addition $\lambda^2 + \mu^2 = k^2$, where k is defined in Eq. (13), then $f(x,y) \in \mathcal{K}er(\mathcal{H}_M)$.

Definition 1. *We say that boundary function $\Phi(x,y)$ is compatible with RHS $f(x,y)$ of Eq. (13) with respect to operator $\mathcal{H}_M$ if*

$$\forall p \in \partial\Omega_C, \quad \mathcal{H}_M\big(\Phi(p)\big) = f(p) \tag{17}$$

4.3 Solution of the Modified Helmholtz equation with homogeneous RHS

As an intermediate stage in the solution of Eq. (13), we solve the Modified Helmholtz equation with zero RHS. We are interested in the solution of

$$\begin{cases} \triangle u_0(x,y) - k^2 u_0(x,y) = 0 & \text{in } \Omega \\ u_0(x,y) = \Phi_0(x,y) & \text{on } \partial\Omega \end{cases} \tag{18}$$

The boundary function $\Phi(x,y)$ is assumed to be smooth and compatible with respect to $\mathcal{H}_M$. In order to utilize a rapidly convergent series expansions (see [1]), the boundary functions $\phi_j,\ j \in I$ should vanish at the p_j along with a number of even derivatives. For each function $q_{2r}(x)$ in the form (15), define four functions $Q_{2r,j}(x,y),\ j \in I$ as follows:

$$\begin{aligned} Q_{2r,1}(x,y) &= \sum_{i=1}^{r+1} \alpha_{2r,i}\,\frac{\sinh(\lambda_{2r,i}\,(1-x))}{\sinh(\lambda_{2r,i})}\,\frac{\sinh(\mu_i\,(1-y))}{\sinh(\mu_{2r,i})}, \\ Q_{2r,2}(x,y) &= Q_{2r,1}(x, 1-y), \\ Q_{2r,3}(x,y) &= Q_{2r,1}(1-x, 1-y), \\ Q_{2r,4}(x,y) &= Q_{2r,1}(1-x, y) \end{aligned} \tag{19}$$

where $\forall i:\ \lambda_{2r,i}, \mu_{2r,i} > 0$ and $\lambda_{2r,i}^2 + \mu_{2r,i}^2 = k^2$.
By virtue of Lemma (2), $\forall j \in I:\ Q_{2r,j}(x,y) \in \mathcal{K}er(\mathcal{H}_M)$.

We define $w_0(x,y)$ and $\Phi_2(x,y)$ as follows

$$\begin{aligned} w_0(x,y) &= \phi_1(0)\,Q_{0,1}(x,y) + \phi_3(0)\,Q_{0,2}(x,y) \\ &\quad + \phi_3(1)\,Q_{0,3}(x,y) + \phi_1(1)\,Q_{0,4}(x,y), \\ \Phi_2(x,y) &= \Phi_0(x,y) - w_0(x,y)|_{\partial\Omega} \end{aligned} \tag{20}$$

$\Phi_2(x,y)$ has the following property: $\forall p \in \partial\Omega_C,\ \Phi_2(p) = 0$.

By solving a new equation

$$\begin{cases} \triangle u_2(x,y) - k^2 u_2(x,y) = 0 & \text{in } \Omega \\ u_2(x,y) = \Phi_2(x,y) & \text{on } \partial\Omega \end{cases} \tag{21}$$

we obtain that $u_0(x,y) = u_2(x,y) + w_0(x,y)$.
The subtraction procedure can be continued. In general, for $r \geqslant 1$, we define

$$\begin{aligned} w_{2(r-1)}(x,y) &= \sum_{j\in I} \Phi^{(2r,0)}_{2(r-1)}(x,y)\big|_{P_j}\, Q_{2(r-1),j}(x,y) \\ \Phi_{2r}(x,y) &= \Phi_{2(r-1)}(x,y) - w_{2(r-1)}(x,y)|_{\partial\Omega} \end{aligned} \tag{22}$$

Lemma 3. *For any $r \geqslant 1$ and any s, $0 \leqslant s \leqslant r-1$ the function $\Phi_{2r}(x,y)$ defined in (22) has the following property:*

$$\Phi^{(2s,0)}_{2r}(p) = \Phi^{(0,2s)}_{2r}(p) = 0,\ \forall p \in \partial\Omega_C \tag{23}$$

Thus, by solving

$$\begin{cases} \triangle u_{2r}(x,y) - k^2 u_{2r}(x,y) = 0 & \text{in } \Omega \\ u_{2r}(x,y) = \Phi_{2r}(x,y) & \text{on } \partial\Omega \end{cases} \tag{24}$$

using rapidly convergent series (as suggested in [1]) we can achieve any prescribed (depending on $r \in \mathbb{N}$) order of accuracy. For $r \geqslant 1$, the general formula for the sought solution of Eq. (18) is

$$u_0(x,y) = u_{2r}(x,y) + \sum_{s=0}^{r-1} w_{2s}(x,y) \tag{25}$$

It is worthwhile to mention that all the functions $w_{2s}(x,y),\ 0 \leqslant s \leqslant r-1$ are explicitly known.

4.4 Solution of the Modified Helmholtz equation with nonhomogeneous RHS

We are interested in the solution of

$$\begin{cases} \triangle u_0(x,y) - k^2 u_0(x,y) = f_0(x,y) & \text{in } \Omega \\ u_0(x,y) = \Psi(x,y) & \text{on } \partial\Omega \end{cases} \tag{26}$$

In addition to the assumptions made in (18), we assume that $f_0(x,y)$ is smooth and $\Phi(x,y)$ is compatible with $f(x,y)$ with respect to $\mathcal{H}_M$. We extend further the technique developed in [2]. In order to solve Eq. (26) with high accuracy, $f_0(x,y)$ should satisfy the conditions stated in the next theorem which where obtained in ([2]).

Theorem 1. *Assume $f_0(x,y)$ is smooth and $p \geqslant 2$. If $\forall s,\ 0 \leqslant s \leqslant p-2$*

$$f_0^{(2s,2s)}(x,y) = 0, \quad \forall p \in \partial\Omega \tag{27}$$

then the direct Fourier method applied to (26) with $\Psi(x,y) = 0$ is of order of accuracy $O(N^{-2p})$.

We look for a function $f(x,y)$ that is an eigenfunction of the operator $\mathcal{H}_M$.

Lemma 4. *Let $\lambda, \mu > 0$ and $f(x,y)$ defined as in (16). If in addition $\lambda^2 + \mu^2 = 1 + k^2$, where k is defined in Eq. (13), then $\mathcal{H}_M(f(x,y)) = f(x,y)$.*

Define four functions $\widetilde{Q}_{2r,j}(x,y),\ j \in I$ as follows:

$$\begin{aligned}
\widetilde{Q}_{2r,1}(x,y) &= \sum_{i=1}^{r+1} \alpha_{2r,i} \frac{\sinh(\lambda_{2r,i}\,(1-x))}{\sinh(\lambda_{2r,i})} \frac{\sinh(\mu_i\,(1-y))}{\sinh(\mu_{2r,i})}, \\
\widetilde{Q}_{2r,2}(x,y) &= \widetilde{Q}_{2r,1}(x, 1-y), \\
\widetilde{Q}_{2r,3}(x,y) &= \widetilde{Q}_{2r,1}(1-x, 1-y), \\
\widetilde{Q}_{2r,4}(x,y) &= \widetilde{Q}_{2r,1}(1-x, y)
\end{aligned} \tag{28}$$

where $\forall i:\ \lambda_{2r,i}, \mu_{2r,i} > 0$ and $\lambda_{2r,i}^2 + \mu_{2r,i}^2 = 1 + k^2$.
By virtue of Lemma (4), $\forall j \in I:\ \mathcal{H}_M(\widetilde{Q}_{2r,j}(x,y)) = \widetilde{Q}_{2r,j}(x,y)$.

We split Eq. (26) to two equations one with homogeneous and one withnon-homogeneous R.H.S. The main idea is to solve Eq. (26) with carefully constructed boundary conditions such that we can achieve any prescribed order of accuracy.

Define $h_0(x,y)$ and $f_1(x,y)$ as follows

$$\begin{aligned}
h_0(x,y) &= \sum_{j\in I} f_0(p_j)\, \widetilde{Q}_{0,j}(x,y) \\
f_1(x,y) &= f_0(x,y) - h_0(x,y)
\end{aligned} \tag{29}$$

Obviously, $\mathcal{H}_M(h_0(x,y)) = h_0(x,y)$ and $f_1(x,y)$ has the following property: $f_1(p) = 0, \forall p \in \partial\Omega_C$. In order to apply Theorem 1, we need that $f_1(x,y)$ will vanish on the boundaries, that is: $f_1(p) = 0,\ \forall p \in \partial\Omega_E$.
For $q_0(x)$ as defined in (15), define

$$\begin{aligned}
\widetilde{q}_{0,1}(y) &= q_0(y), & \widetilde{q}_{2r,2}(x) &= q_{2r}(x), \\
\widetilde{q}_{0,3}(y) &= q_0(1-y), & \widetilde{q}_{2r,4}(x) &= q_{2r}(1-x)
\end{aligned}$$

and also (where $\zeta \equiv y$ for $j = 1,3$ and $\zeta \equiv x$ for $j = 2,4$)

$$h_{1,j}(x,y) = f_1(x,y)|_{E_j}\ \widetilde{q}_{0,j}(\zeta), \quad h_1(x,y) = \sum_{j\in I} h_{1,j}(x,y),$$

$$f_2(x,y) \triangleq f_1(x,y) - h_1(x,y)$$

We introduce the following problems:

$$\forall j \in I : \begin{cases} \triangle w_{0,j}(x,y) - k^2 w_{0,j}(x,y) = h_1(x,y) & \text{in } \Omega \\ w_{0,j}(x,y) = 0 & \text{on } \partial\Omega \end{cases}$$
$$\begin{cases} \triangle u_2(x,y) - k^2 u_2(x,y) = f_2(x,y) & \text{in } \Omega \\ u_2(x,y) = 0 & \text{on } \partial\Omega \end{cases} \tag{30}$$

Using the technique of [2] for the error estimates, it can be shown that each equation in (30) can be solved with $O(N^{-4})$ order of accuracy and therefore, Eq. (26) with $\widetilde{\Psi}(x,y) = h_0(x,y)|_{\partial\Omega}$ can be also solved with $O(N^{-4})$ accuracy. In addition, we need to solve Eq. (18) with $\Phi(x,y) = \Psi(x,y) - \widetilde{\Psi}(x,y)$. We can proceed further and by constructing $h_2(x,y)$ and $h_3(x,y)$ obtain $O(N^{-6})$ accuracy etc..

4.5 Solution of the Modified Helmholtz Equation in the Non-Compatible Case

In the formulation of the original problem, the boundary function $\Phi(x,y)$ is not necessary compatible with the R.H.S. with respect to $\mathcal{H}_M$. We utilize the idea that by changing the boundary function $\Phi(x,y)$ along with $f(x,y)$ in (13) we can achieve compatibility of the boundary function and the R.H.S. For this purpose we can use functions of the form

$$\tau_{2k}(x,y) = \mathcal{R}e\{c_{2k} z^{2k} log(z)\} \tag{31}$$

where $c_{2k} = a_{2k} + ib_{2k}$ and where $a_{2k} = 0$ while $b_{2k} = (-1)^k \dfrac{2}{\pi(2k)!}$.
As an example, assume that compatibility doesn't hold at p_1, that is:

$$\phi_1''(0) + \phi_2''(0) - k^2\phi_1(0) = f(p_1) + A$$

Let $v(x,y) = u(x,y) - A\tau_2(x,y)$. For $v(x,y)$ compatibility holds at p_1. Also, assume that compatibility already holds for the other corners. Thus, if we define

$$\widetilde{f}(x,y) \triangleq f(x,y) + Ak^2\tau_2(x,y)$$
$$\widetilde{\Phi}(x,y) \triangleq \Phi(x,y) - A\tau_2(x,y)|_{\partial\Omega}$$

then for

$$\begin{cases} \triangle v(x,y) - k^2 v(x,y) = \widetilde{f}(x,y) & \text{in } \Omega \\ v(x,y) = \widetilde{\Phi}(x,y) & \text{on } \partial\Omega \end{cases} \tag{32}$$

compatibility of $\widetilde{\Phi}(x,y)$ with $\widetilde{f}(x,y)$ holds. After solution of Eq. (32) we return back to $u(x,y)$.

5 Domain Decomposition

The present algorithm incorporates the following novel elements:

1. It extends our previous fast Poisson solvers [**?**] as it provides an essentially direct solution for equations (2) where $a(x,y)^{1/2}$ is an arbitrary harmonic function.
2. In the case where $a(x,y)^{1/2}$ is not harmonic, we approximate it by $\widetilde{a}(x,y)^{1/2}$ (which is a superposition of harmonic functions) and apply several correction steps to improve the accuracy.
3. In the case where $a(x,y)^{1/2}$ is dome shaped or bowl shaped, we approximate it by $\widetilde{a}(x,y)^{1/2}$ which is now a solution of Eq. (**??**) and apply several correction steps to improve the accuracy. The value of P is determined to match the average Gaussian curvature of $a(x,y)^{1/2}$

However high accuracy for the solution of (2) requires an accurate approximation of $a(x,y)^{1/2}$ by the functions discussed above. Such an approximation is not always easy to derive in the global domain, however it can be achieved readily in smaller subdomains. In this case we suggest the following Domain Decomposition algorithm.

1. The domain is decomposed into smaller rectangular subdomains. Where the boundary of the subdomains coincides with full domain boundary we take on the original boundary conditions. For other interfaces we introduce some initial boundary conditions which do not contradict the equation at the corners, where the left hand side of (2) can be computed. The function $a(x,y)$ is approximated by $\tilde{a}(x,y)^{1/2}$ in each subdomain such that $\tilde{a}(x,y)^{1/2}$ is harmonic(or subharmonic or superharmonic). An auxiliary equation (12) is solved in each subdomain.
2. The collection of solutions obtained at Step 1 is continuous but doesn't have continuous derivatives at domain interfaces. To further match subdomains, a hierarchical procedure can be applied similar to the one described in [4]. For example, if we have four subdomains 1,2,3 and 4, then 1 can be matched with 2, 3 with 4, while at the final step the merged domain 1,2 is matched with 3,4.

We illustrate the effectiveness of the domain decomposition approach by solving a one dimensional variable coefficient equation where the coefficient function is not harmonic namely $a(x) = (2x + 3 + \sin(\pi x))^2$ with exact solution $u(x) = \sin(\pi x)$. We change the number of domains from 1 to 8. The correction procedure works much better when the subdomains become smaller. With 4 domains and with only 2 correction steps we reach an error of order 10^{-6}, with 8 domains we get 10^{-8}. Thus the present approach behaves essentially as a direct fast method. The Domain Decomposition of course has the further advantage of easy parallelization on massively parallel computers.

6 Numerical results

First let us demonstrate the rate of convergence of the improved subtraction algorithm.Assume that u is the exact solution of Eq. (2) and u' is the computed solution. We will use the following measure to estimate the errors:

$$\varepsilon_{MAX} = \max |u'_i - u_i| \quad (33)$$

Example 1. Consider the Modified Helmholtz equation with $f(x,y) = -k^2(x^2 - y^2)$, where k is defined in (13); the right hand side and the boundary conditions correspond to the exact solution $u(x,y) = x^2 - y^2$ in the domain $[0,1] \times [0,1]$. The results are presented in Table 1.

$N_x \times N_y$	$\varepsilon_{MAX}(4)$	ratio
8×8	1.29e-6	-
16×16	1.23e-7	10.5
32×32	9.81e-9	12.5
64×64	7.04e-10	13.93
128×64	4.81e-11	14.66

Table 1: MAX error for the fourth order subtraction methods with $k = 1$.

Example 2. Consider the same equation as in the previous example but with $k = 10$. The results are presented in Table 2.

$N_x \times N_y$	$\varepsilon_{MAX}(4)$	ratio
8×8	3.38e-3	-
16×16	2.89e-4	11.7
32×32	2.73e-5	10.58
64×64	2.16e-6	12.64
128×64	1.54e-7	14.02

Table 2: MAX error for the fourth order subtraction methods with $k = 10$.

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