A Particle Gradient Evolutionary Algorithm Based on Statistical Mechanics and Convergence Analysis

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Abstract. In this paper a particle gradient evolutionary algorithm is presented for solving complex single-objective optimization problems based on statistical mechanics theory, the principle of gradient descending and the law of evolving chance ascending of particles. Numerical experiments show that we can easily solve complex single-objective optimization problems that are difficult to solve by using traditional evolutionary algorithms and avoid the premature phenomenon of these problems. In addition, a convergence analysis of the algorithm indicates that it can quickly converge to optimal solutions of the optimization problems. Hence this algorithm is more reliable and stable than traditional evolutionary algorithms.

1 Introduction

Evolutionary algorithms (EAs) are searching methods that take their inspiration from natural selection and survival of the fittest in the biological world [1,2]. EAs differ from traditional optimization techniques in that they involve a search from a "population" of solutions, not from a single point. Each iteration of an EA involves a competitive selection that weeds out poor solutions. The solutions with high "fitness" are "recombined" with other solutions by crossing parts of a solution with another. Solutions are also "mutated" by making a small change to a single element of the solutions. Recombination and mutation are used to generate new solutions that are biased toward regions of the space for which good solutions have already been seen. However, there are two main problems puzzling researches in the literature of EC (evolutionary computation) research. The first is the premature, which is one of the basic problems in EC research, and the second is the lack of a proper stopping criterion in problem solution.

Previous evolutionary algorithms (we call them traditional evolutionary algorithms) are difficult to avoid the premature phenomenon, and fall into local optimal solutions; the reason is that the traditional evolutionary algorithms cannot take all the individuals of population to participate in crossing and mutating all the time.

In this paper a particle gradient evolutionary algorithm for solving complex single-objective optimization problems (SPGEA) is presented to overcome the shortcomings of the traditional evolutionary algorithms mentioned above. SPGEA adopts the method of solving the gradient of an optimization problem to construct the fitness function of the problem, which simulates the principle of energy minimizing of particles in statistical mechanics, and designs an evolving chance function of individuals as the amount of individual crossing, which simulates the law of entropy increasing of particles in statistical mechanics. Based on this construction method, the algorithm guarantees that all the particles have a chance to cross and evolve all the time and produces the global optimization solution of a problem.

This paper is organized as follows: In Section 2, theoretical foundations of statistical mechanics are discussed. The principle of gradient descending and the law of evolving chance ascending in a particle system are then analyzed theoretically in Section 3. A detailed description of a SPGEA flow is designed in Section 4. In Section 5, we perform experiments to test SPGEA by solving three complex optimization problems. The convergence of SPGEA is studied in Section 6. Finally, we draw some conclusions in Section 7.

2 Relevant Theories of Statistical Mechanics

Statistical mechanics [3,4] is to apply a statistical analysis method of applied mathematics to study the average behavior and statistical rules of a number of particles. It is an important branch of theoretical physics. The non-equilibrium statistical mechanics is to study more complex problems. Not until in the mid-20th century has the study of statistical mechanics achieved a rapid development. For a macro physical system being composed of a number of particles, the probability of the system that keeps a more disordered state exceeds the probability of the system that keeps a more ordered state. A closed physical system always trends to the disordered state from the ordered state. In thermodynamics, this is the corresponding law of entropy ascending. Therefore, the free energy theory and entropy theory of statistical mechanics are very important in the course of discussing the equilibrium and non-equilibrium particle system below.

2.1 Law of Entropy Ascending

Assume that a closed system is composed of two open subsystems that may exchange energy and particles so that the entropy of the system increases, i.e., $S = S_1 + S_2$, where S_1 and S_2 denote the entropies of the first and second systems, respectively. Furthermore, assume that the relationship between the micro-state

number of the micro-canonical ensemble and the entropy function is $S = f(\Delta\Omega)$, and the two subsystems are independent of each other. As a result, the micro-state number of an isolated system is $\Delta\Omega = \Delta\Omega_1\Delta\Omega_2$. Thus $S_1 + S_2 = f(\Delta\Omega_1\Delta\Omega_2)$ and $S = k_B \ln\Delta\Omega$, where k_B is called the Boltzmann constant. According to the entropy equilibrium equation and Boltzman H-theorem, we see that the entropy function is a monotonically increasing function of time in a closed system; i.e., $\frac{dS(t)}{dt} \geq 0$. Therefore, the entropy is irreversible in the thermo-insulated system, which is the law of entropy increasing.

2.2 Principle of Energy Descending

The concept of "free energy" is a key concept to characterize physically relevant states in statistical mechanics. Given an equilibrium system of statistical mechanics with energy levels E_i of the microstates i, the Helmholtz free energy is defined as

$$F(b) = -\frac{1}{b}\log Z(b),$$

where

$$Z(b) = \sum_i e^{-bE_i}$$

is the partition function and b is the inverse temperature. Apparently, the Helmholtz free energy is different from the internal energy U given by

$$U = -\frac{\partial}{\partial b}\log Z(b) = \langle E_i \rangle.$$

The difference is given by the entropy times the temperature:

$$F = U - TS.$$

This equation can also be regarded as descending a Legendre transformation from U to F. Equilibrium states minimize the free energy; in this sense F is more relevant than U. The minimum of F can be achieved in two competing ways: Either by making the internal energy U small or by making the entropy S large. The basic principle underlying statistical mechanics, the maximum entropy principle, can also be formulated as a "principle of minimum free energy".

Through the above analysis of a particle system, we know that the equilibrium state of the particle system depends on the result of the competition between free energy descending of this particle system and entropy ascending.

3 Principle of Gradient Descending and Law of Evolving Chance Ascending

We apply the principle of free energy descending and the law of entropy ascending in statistical mechanics to the SPGEA design. In the design of SPGEA , we consider individuals of a population as particles in the particle phase space, and the population of each generation as a system of particles. Our purpose is to simulate the particle system discipline in the physics system to cross and mutate individuals of the population, which tries to change its state from non-equilibrium to equilibrium, and as a result, solves for all the optimal solutions, and avoid problems' premature.

Because the establishment of a fitness function and an iterative stopping criterion of SPGEA is based on the principle of gradient descending and the law of evolving chance ascending in a physical system, which simulates the law of entropy ascending and the principle of energy descending, SPGEA is guaranteed to drive all the particles in the phase space to participate in crossing and mutating, and to speed up its convergence; in the meantime it improves its computing performance so that the probability of the phase space equals and the equilibrium state in the phase space is achieved.

4 Algorithm Flow of SPGEA

4.1 Description of Optimization Problem

We consider the optimization problem :

$$\min_{X\in D} f(X)\ ,\quad D=\{X\in S; g_k(X)\le 0, k=1,2,\cdots,q\ ,$$

where $S\subset R^n$ is the searching space, usually a hypercube of N dimensions, namely, $l_i\le x_i\le u_i, i=1,2,\cdots,n$, $f: S\to R$ the objective function, n the dimension of the decision space, and D the set of feasible points.

4.2 Variation of the Objective Function

We assume that the population size is N, and the individuals $\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_N$ as N particles in a physical system. Then we add the number t of a continuous evolving iteration into the objective function of the optimization problem, and get the new dynamical single-objective function of the optimization problem $\min_{X\in D,t>0} f(t,X)$ related to the iteration time. We say that $\min_{X\in D,t>0} f(t,X)$ is a SPGEA objective function.

4.3 Algorithm Process of SPGEA

According to the principle of free energy descending and the law of entropy ascending of the physical system, we give the definitions of a gradient descending equation and a evolving chance ascending equation of SPGEA as follows:

Definition 1 (SPGEA gradient descending equation): We call the difference equation $\nabla p(t, \mathbf{x_i}) = \nabla f(t, \mathbf{x_i}) - \nabla f(t-1, \mathbf{x_i})$ as a SPGEA gradient descending equation (SPGEA free energy) of ith particle $\mathbf{x}_i$ at time t, where $f(\mathbf{x})$ is a function on D, $\mathbf{x_i} \in D$, $i = 1,2,3\cdots\cdots$.

Definition 2 (SPGEA evolving chance ascending): We call the evolving chance counting function $a(t, \mathbf{x_i})$ of ith particle $\mathbf{x}_i$ at time t the SPGEA evolving chance ascending (SPGEA entropy), whose value is determined as follows: When particles $\mathbf{x_i}$ participate in the evolving operation in time t,

$$a(t, \mathbf{x_i}) = a(t-1, \mathbf{x_i}) + 1,$$

Otherwise,

$$a(t, \mathbf{x_i}) = a(t-1, \mathbf{x_i}) \text{ , } \mathbf{x_i} \in D, \quad i = 1,2,3\cdots\cdots \quad N \text{。}$$

Definition 3 (SPGEA fitness function): We define the weighted function $select(t, \mathbf{x_i}) = l_1 \sum_{k=0}^{t} \left\| \nabla p(k, \mathbf{x_i}) \right\|_p + l_2 \ln(a(t, \mathbf{x_i}) + 1)$ as the SPGEA fitness function, where $l_1, l_2 \in [0,1]$, $l_1 + l_2 = 1$, and l_1, l_2 are called SPGEA Boltzmann constants, whose values depend on the significance of $\sum_{k=0}^{t} \left\| \nabla p(k, \mathbf{x_i}) \right\|_p$ and $\ln(a(t, \mathbf{x_i}) + 1)$ on the right-hand side of the fitness function equation, respectively. That is, the more significant it is, the larger the corresponding SPGEA Boltzmann constant is. This ensures the whole physical system to reach the equilibrium state from the non-equilibrium state, and hence to achieve the equal probability in the phase space; in the meantime, all the individuals in the population have a chance to take part in crossing and mutating at all the iteration times so that global optimal solutions can be achieved. In the SPGEA fitness function we can also see that the reason why SPGEA can avoid the premature phenomenon is that the SPGEA fitness function contains the SPGEA gradient descending term (SPGEA free energy) and the SPGEA evolving chance ascending term (SPGEA entropy).

Definition 4 (SPGEA stopping criterion): We define a SPGEA stopping criterion by

$$(\sum_{i=1}^{N} \left\| \nabla p(t, \mathbf{x_i}) \right\|_p) / t < e \text{ or } \sum_{i=1}^{N} \ln(a(t, \mathbf{x_i}) + 1) > T \text{ ,}$$

where e is a given small positive constant.

The first SPGEA stopping criterion is constructed by SPGEA free energy, and the second SPGEA stopping criterion is built by SPGEA entropy. We can easily see that the purpose of SPGEA is to minimize SPGEA free energy and maximize SPGEA entropy. These two terms are like the Helmholtz free energy and the entropy of particles

in the physical system, and always compete with each other in the course of changing from non-equilibrium to equilibrium spontaneously under the same temperature.

According to the above four basic definitions of SPGEA we design the detailed algorithm of SPGEA as follows:

Step 1: Initialize particles in the physical system to generate an initial population with N individuals $\Gamma_N = \{\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_N}\}$ randomly, and set $t := 0$.

Step 2: Calculate all the function values of the particles in Γ_N and set $\nabla p(t, \mathbf{x_i}) = 0$, $a(t, \mathbf{x_i}) = 0$, $\mathbf{x_i} \in \Gamma_N$; then calculate the fitness values of fitness functions $select(t, \mathbf{x_i})$, which are in the order from small to large.

Step 3: Save all the particles and their function values in the system Γ_N.

Step 4: Begin to iterate: $t =: t+1$.

Step 5: Select n particles $\mathbf{x}_i'$, $i = 1,2,\cdots,n$ on the forefront of $select(t-1, \mathbf{x_i})$; if all the values of $select(t-1, \mathbf{x_i})$ are the same, select n particles randomly.

Step 6: Implement evolving operations on the n particles of the physical system, and generate n random numbers $a_i \in [-1,1], i = 1,2,\cdots,n$ that satisfy $-0.5 \le \sum_{i=1}^{n} a_i \le 1.5$ and $\hat{\mathbf{x}} = \sum_{i=1}^{n} a_i \mathbf{x}_i' \in \mathbf{X}$; if the function value at the point $\hat{\mathbf{x}}$ is better than the worst function value at the point $\tilde{\mathbf{x}}_i$, then we replace the individual $\tilde{\mathbf{x}}_i$ by $\hat{\mathbf{x}}$; otherwise repeat this evolving operation.

Step 7: Save the best particles, and their function values and fitness values in the system Γ_N.

Step 8: Renew all the values of $select(t, \mathbf{x_i})$ and re-sort in an ascending order.

Step 9: Calculate the stopping criterion; if $(\sum_{i=1}^{N} \|\nabla p(t, \mathbf{x_i})\|_p) / t < e$ or $\sum_{i=1}^{N} \ln(a(t, \mathbf{x_i}) + 1) > T$, stop iteration; otherwise, go to step 4.

In the above two SPGEA stopping criteria we know that the individuals which are not selected in the previous generation have more chance to be selected to take part in the evolving operation in the next generation because the fitness values added up by non-selected individuals in the previous generation are less than other individuals' fitness values calculated by the selected individuals in the previous generation. In this way, it is guaranteed that all the individuals in the population have a chance to take part in crossing and mutating all the time; this is one of the main features of SPGEA.

5 Data Experiments

In this section, three typical optimization problems that are difficult to solve using the traditional EA [5-8] will be experimented to test the performance of SPGEA. In the first experiment, we use SPGEA to solve the minimization problem of the function

$$\min_{x \in S} f(x_1, x_2) = x_1^2 + x_2^2 - 0.3\cos(3\pi x_1) - 0.4\cos(4\pi x_2) + 0.7,$$

where $-50 \le x_1 \le 50$ and $-50 \le x_2 \le 50$. From Fig.1 we can see that this optimization problem has almost an infinite number of local optimal points in the searching space, but there is only one global minimum point at $x_1 = 0$ and $x_2 = 0$ that reaches the minimum value $f^* = 0$ of the function. Only the local optimal points can be solved by using the traditional evolutionary algorithm in general.

In this experiment, we set the population size $N = 80$. The weighted coefficient $l_1 = 0.8, l_2 = 0.2, e = 10^{-11}$, and the maximal value of the evolving chance function $T = 10^8$, and then select four particles (individuals) that are located in front of the fitness values of the function $select(t, \mathbf{x}_i)$ in the order from small to large to cross and mutate. According to the above configured parameters we run the SPGEA program 10 times continuously, in every iteration time we can get the optimal point that is given in Table 1 The convergent speed by using SPGEA is faster and the results are more accurate than the traditional evolutionary algorithm in Ref. [8].

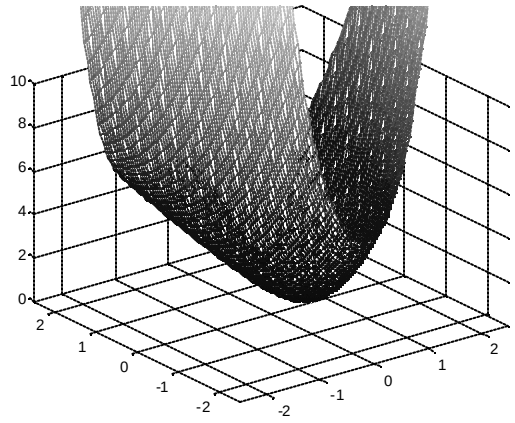

Fig.1 the landscape of experiment 1

Table1: The results of running SPGEA program 10 times in experiment 1

Min value f	x_1	x_2	step
0	1.53257e-010	-1.92252e-009	796
0	2.20815e-010	9.12350e-012	780
0	-1.13563e-009	-4.89725e-010	696
0	-1.92390e-009	1.30710e-010	718
0	3.66944e-010	6.67918e-010	708
0	-1.63819e-010	2.56498e-010	670
0	-2.59710e-010	1.85425e-011	831
0	1.02611e-009	-3.04113e-010	747
0	-5.06763e-011	5.91274e-010	745
0	4.64298e-010	2.06257e-010	812

In the second experiment, we use SPGEA to test a non-convex function as follows:

$$\min_{\mathbf{x}\in S} f(x_1, x_2) = 100(x_1^2 - x_2)^2 + (1 - x_1)^2$$

where $-2.048 \le x_1 \le 2.048, -2.048 \le x_2 \le 2.048$; this function is non-convex (see Fig.2). In running SPGEA, we set the population size $N = 60$, $l_1 = 0.8, l_2 = 0.2, e = 10^{-20}$, and the maximal value of the evolving chance function $T = 10^6$; the crossing and mutating method is the same as in the first experiment. Running the SPGEA program 10 times continuously, we can get the optimal point every time which is given in Table 2.

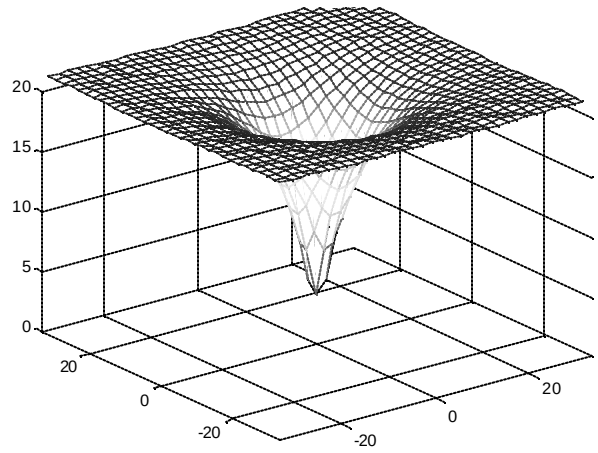

Fig.2 the landscape of experiment 2

Table2: The results of running SPGEA program 10 times in experiment 2

Min value f	x_1	x_2	step
3.11543E-244	1.00000E+00	1.00000E+00	9976
1.06561E-241	1.00000E+00	1.00000E+00	9999
9.01807E-241	1.00000E+00	1.00000E+00	10000
7.33725E-243	1.00000E+00	1.00000E+00	9991
8.85133E-243	1.00000E+00	1.00000E+00	9984
1.75654E-243	1.00000E+00	1.00000E+00	9989
4.45935E-242	1.00000E+00	1.00000E+00	9999
5.99649E-242	1.00000E+00	1.00000E+00	9998

In the third experiment, a complex single-objective minimization problem (Ackley function) is tested by using SPGEA, and the optimization problem is as follows:

$$\min_{\mathbf{x}\in S} f(x_1, x_2) = -20\exp\left(-0.2\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right) - \exp\left(\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)\right) + 20 + e$$

where $-32.768 \le x_i \le 32.768, i = 1,2,\cdots n$, $n = 2$. It is obvious that the optimal solution $f^* = 0$ is reached at the point $x_1 = 0$ and $x_2 = 0$. From Fig.3 we can also see that

this function is non-convex and includes multi-local-optimal-points. So it is difficult to solve by using the traditional evolutionary algorithms. In fact, it is very easy to fall into the local solutions, i.e., the premature phenomenon of the algorithms.

We set the parameters of SPGEA: population size $N = 80$. $l_1 = 0.8, l_2 = 0.2, e = 10^{-20}$, and the maximal value of the evolving chance function $T = 10^6$; the crossing and mutating method is same as in the first experiment. Running the SPGEA program 10 times continuously, in each iteration time we also can easily get the optimal point that is given in Table 3.

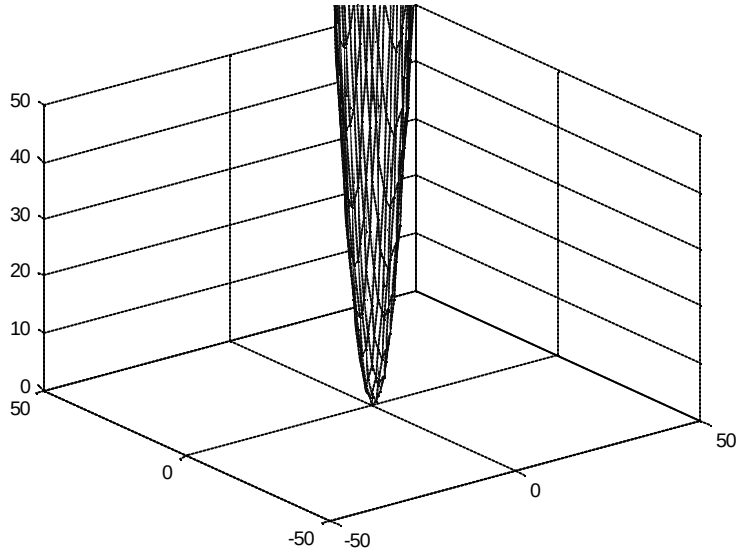

Fig.3 the landscape of experiment 3

Table3: The results of running SPGEA program 10 times in experiment 3

Min value f	x_1	x_2	step
1.54096e-009	1.23727e-016	2.74191e-016	913
1.54096e-009	-1.03297e-016	-7.22339e-017	975
1.54096e-009	-3.43706e-017	-8.11170e-017	855
1.54096e-009	2.35071e-016	1.00479e-016	777
1.54096e-009	2.75004e-017	2.20330e-017	871
1.54096e-009	-2.01528e-016	1.24738e-016	780
1.54096e-009	1.66060e-017	1.26505e-017	893
1.54096e-009	-7.42641e-019	7.21961e-019	822
1.54096e-009	-4.29195e-016	-1.45410e-016	938
1.54096e-009	-4.87201e-017	-1.01773e-017	902

Furthermore, we have also done many experiments to solve some well-konwn complex single-objective optimization problems like the Six Hump Camel Back Function,

Axis-Parallel Hyperellipsoid Function, and Griewangk's Function by using SPGEA, and we have obtained very accurate optimal solutions, which are difficult to solve by using the traditional evolutionary algorithms.

6 Convergent Analysis of SPGEA

The convergence, time efficiency, and precision of optimal solutions are very important factors when optimization problems are solved by using evolutionary algorithms [9-12], which provide a reliable indication to an efficient algorithm. In this section the convergence of SPGEA is studied to illustrate the advantages of SPGEA according to the theory of the Markov chain and other convergent theories related to EC [13-15].

Assume the optimization problem:

$$\min\{f(x) \mid x \in S\}, \qquad (1)$$

where f is a function in the decision space S and $\forall \mathbf{x} \in S, f(\mathbf{x}) \geq 0$. S can be either a finite set (e.g., composition optimization problems), or a set in the real space R^n (e.g., continuous optimization problems). Then we get four definitions as follows:

Definition 5: For the SPGEA optimization problem (1), suppose that random variable $T = \min\{t \in \overline{Z}^- : F_t = f^* : \overline{Z}^- = \{0,1,\cdots,n,\cdots\}\}$ represents the time the global optimal point is found at the first time, if $P\{T < \infty\} = 1$ and independent of the initial population; then we say that the SPGEA algorithm can find the global optimal solutions of optimization problems in probability 1 in the finite time [16].

Definition 6: For a non-negative random variable sequence $\{X_n\}$, $t = 0,1,\cdots,\cdots$, which is defined in the probability space (Ω, A, P).

(1) If $\forall e > 0$ such that $\sum_{n=0}^{\infty} P(X_n > e)$ is convergent, then $\{X_n\}$ is called completely convergent to 0.

(2) If $\forall e > 0$ such that $\sum_{n=0}^{\infty} P\{w : \lim_{n\to\infty} X_n(w) = 0\} = 1$, then $\{X_n\}$ is called convergent to 0 in the probability 1.

(3) If $\forall e > 0$ such that $\lim_{n\to\infty} P\{X_n(w) > e\} = 0$, then $\{X_n\}$ is called convergent to 0 in probability.

In the above three convergent forms, the completely convergent is the strongest, which implies both the convergent in the probability 1 and the convergent in probability, and the convergent in probability is the weakest [16].

Define $D_t = d(X_t) = F_t - f^*$, where f^* is the optimal solution of the optimization problem, and $F_t = f(\mathbf{x}_t)$ is the best solution of the optimization problem in the t th generation, and then set the convergence definition of SPGEA as follows:

Definition 7: We call solving optimization problem (1) the completely convergent (the convergent in the probability 1 or the convergent in probability) to the global optimal points of the problem, if the non-negative random sequence $(D_t : t \geq 0)$ pro-

duced by this optimization problem is completely convergent (convergent in the probability 1 or convergent in probability) to 0 [16].

Definition 8: We call solving optimization problem (1) which adopts the elite reservation strategy by using SPGEA the completely convergent (the convergent in the probability 1 or the convergent in probability) to the global optimal point f^*, if the non-negative random sequence $(D_t : t \geq 0)$ produced by this optimization problem is completely convergent (convergent in the probability 1 or convergent in probability) to 0 [16].

According to the above definitions, we get the next convergence theories of SPGEA.

Theorem 1: If the optimization problem (1) solved by using SPGEA satisfies the following conditions:

(1) In every evolving iteration t, if $\mathbf{x} \neq \tilde{\mathbf{x}}$ for all individuals $\mathbf{x}$ $(\mathbf{x} \in P(t))$ in the population $P(t)$ and $\forall \tilde{\mathbf{x}} \in S$, then through crossover and mutation operation once, the probability mutating $\mathbf{x}$ to $\tilde{\mathbf{x}}$ is more than or equal to $p(t)$, where $p(t)$ is a constant more than 0, and the probability is related to generation t.

(2) $\prod_{t=1}^{\infty}(1-p(t)) = 0$.

Then SPGEA can certainly find the global optimal solution of the optimization problem in probability in finite generation times, that is,

$P\{T < \infty\} = 1$, and it has nothing to do with the distribution of the iterating initial population.

Proof: In the evolving process of tth generation, it needs to mutate to N individuals of the population by using a mutation operator independently from condition (1); we know that in tth generation of any evolving operation, through the mutation of the mutating operator, any individual in population $P(t)$ can mutate to any other individual in the search space S on the lower boundary probability $p(t)$. Therefore, in the mutation process the probability which mutates any individual $\mathbf{x} \notin \arg f^*$ to one of the global optimal solutions is no less than $p(t)$, i.e. the probability which is the first found global optimal point is at least $p(t)(> 0)$ in the evolving process of t th generation. So, after the t generations, the probability $\overline{p}(t)$ that no global optimal point found satisfies

$$\overline{p}(t) \leq \prod_{i=1}^{t}(1-p(t)),$$

namely, $\lim_{t\to\infty} \overline{p}(t) = 0$, and then we get

$$P\{T < \infty\} \geq P\{\text{find a global optimal point in } t \text{ generations}\} = 1 - \overline{p}(t).$$

Setting $t \to \infty$ on both sides of the above equation, it reduce to $P\{T < \infty\} = 1$; that is, SPGEA can find global optimal solutions of the optimization problem in the probability 1 in the finite evolving times, and obviously, from the proof process we can

see that it has nothing to do with the selecting method of an initial population in this theorem's proof.

Theorem 2: If the optimization problem (1) solved by using SPGEA satisfies the following conditions:

(1) In every evolving iteration t, if $\mathbf{x} \neq \mathbf{y}$ for all individuals $\mathbf{x}$ ($\mathbf{x} \in P(t)$) in the population $P(t)$ and $\forall \mathbf{y} \in S$, then by crossover and mutation operation once, the probability mutating $\mathbf{x}$ to $\mathbf{y}$ is more than or equal to $p(t)$, where $p(t)$ is a constant more than 0 and the probability is related to generation t.

(2) $\prod_{t=1}^{\infty}(1-p(t))=0$.

(3) Adopt the strategy of the elite reservation to evolve.

Then, SPGEA certainly converges to the optimal solution of the optimization problem in probability, and it has nothing to do with the selecting method of an initial population.

If SPGEA satisfies the following additional condition:

(4) There exists a constant $p>0$ such that $p(t) \geq p$ for all generation t, then SPGEA is completely convergent.

Proof: Assume that the global optimal point of the optimization problem is first found in the tth generation, because SPGEA is evolved according to the strategy of elite reservation. This guarantees the first found optimal solution individual to be maintained ever to the last generation in the evolving. Hence we get

$$P\{F_t > f^*\} = P\{F_t - f^* > 0\} = P\{D_t > 0\} = \prod_{i=1}^{t}(1-p(i)).$$

Setting $t \to \infty$, it is

$\lim_{t\to\infty} P\{D_t > 0\} = 0$; namely, SPGEA is convergent in probability.

If SPGEA satisfies condition (4), too, e.g., if we can find a constant $p>0$ such that $p(t) \geq p$ for all t, then

$$P\{F_t > f^*\} = P\{F_t - f^* > 0\} = P\{D_t > 0\} = \prod_{i=1}^{t}(1-p(i))$$
$$\leq \prod_{i=1}^{t}(1-p) = (1-p)^t$$

Because the Taylor series $\sum_{n=0}^{\infty}(1-p)^t$ is convergent, according to definitions 6 (1) and 7, we conclude that $(D_t : t \geq 0)$ is completely convergent to 0; accordingly, SPGEA is completely convergent to 0 as well. From all the proof process we can see that the convergence of SPGEA has nothing to do with the selection method of an initial population.

7 Conclusions

Through the above theoretical and experimental analysis of SPGEA, we conclude that SPGEA has obviously more advantages than traditional EAs. Because SPGEA is based on statistical mechanics theory according to the principle of gradient descending and the law of evolving chance ascending of particles, which simulate the principle of energy minimizing and law of entropy increasing in the phase space of particles in statistical mechanics, it makes all the particles to have a chance to evolve, and drives all the particles to cross and mutate to reproduce new individuals of the next generation from the beginning to the end. Because of these reasons SPGEA can easily and quickly search for the global optimal solutions and avoid premature phenomenon of the algorithm. Meanwhile, convergent analysis of SPGEA has proved that it is reliable, stable, and secure by using SPGEA to solve complex single-objective optimization problems.

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